

The University of Chicago

TWO SINGULARITIES OF SPACE CURVES

A PART OF A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES IN
CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1942

By
ALICE TURNER SCHAFFER

Reprinted from
DUKE MATHEMATICAL JOURNAL
Vol. 11, No. 4, December, 1944

The University of Chicago

TWO SINGULARITIES OF SPACE CURVES

A PART OF A DISSERTATION SUBMITTED TO THE FACULTY
OF THE DIVISION OF THE PHYSICAL SCIENCES IN
CANDIDACY FOR THE DEGREE OF DOCTOR OF PHILOSOPHY

DEPARTMENT OF MATHEMATICS
1942

By
ALICE TURNER SCHAFER

Reprinted from
DUKE MATHEMATICAL JOURNAL
Vol. 11, No. 4, December, 1944



TWO SINGULARITIES OF SPACE CURVES

BY ALICE T. SCHAFER

Introduction. The two singularities studied in this paper are an inflexion point and a planar point. In the neighborhood of each of these points the curve can be represented by power series just as is the case in the neighborhood of an ordinary point on a space curve. However, an inflexion point is an exceptional point in the sense that at such a point the tangent of the curve has precisely three-point, instead of the usual two-point, contact with the curve. Similarly, a planar point is an exceptional point in that at such a point the osculating plane of the curve has precisely four-point, instead of the usual three-point, contact with the curve.

In the case of each of these singularities the projective coordinate system is chosen so as to simplify as much as possible the power series representing the curve in the neighborhood of the singularity; that is, to reduce the series to a canonical form. There are fifteen arbitrary parameters involved in the determination of the coordinate system and therefore fifteen conditions which may be imposed on the constants in the power series in deducing the canonical expansions.

Throughout this discussion homogeneous and non-homogeneous coordinates will be used interchangeably. Homogeneous coordinates of a point will be denoted by x_1, x_2, x_3, x_4 and non-homogeneous coordinates by x, y, z . These coordinates are connected by the relations $x = x_2/x_1, y = x_3/x_1, z = x_4/x_1$.

In the first chapter the inflexion point is considered. Canonical power-series expansions are developed in §1.1. The next three sections contain applications of these expansions: §1.2 is a study of some of the surfaces osculating the curve C at the inflexion point O ; in §1.3 sections of the tangent developable of C in the neighborhood of O are studied; and finally, §1.4 is devoted to an investigation of the nature of some of the projections of C .

The second chapter discusses the planar point. The same organization of sections is used as in the case of the inflexion point.

Chapter I. Inflexion Point

1.1. Canonical power-series expansions. Let us consider in ordinary space an analytic curve C which can be represented in a sufficiently small neighborhood of a point $O (c_0, a_0, b_0)$ on C by two power series of the form

$$(1) \quad \begin{aligned} y &= a_0 + a_1(x - c_0) + a_2(x - c_0)^2 + \cdots, \\ z &= b_0 + b_1(x - c_0) + b_2(x - c_0)^2 + \cdots, \end{aligned}$$

Received March 31, 1944; presented to the American Mathematical Society November 26, 1943.

where the coefficients a_i and b_i ($i = 0, 1, 2, \dots$) are constants. In this section the hypothesis will be made that O is an inflexion point on C and the purpose will then be to prove the following theorem:

THEOREM 1.1. *By a suitable choice of the projective coordinate system the power-series expansions representing any space curve C in the neighborhood of an inflexion point O (not otherwise singular) on C can be reduced to the following canonical form:*

$$(2) \quad \begin{aligned} y &= ax^3 - a^2x^5 + a^2(a+1)x^7 + a_8x^8 + a_9x^9 + \dots, \\ z &= ax^4 - a^2x^6 + a^2(a+1)x^8 + b_9x^9 + \dots \quad (a \neq 0). \end{aligned}$$

In order to prove this theorem we shall begin by taking the vertex $(1, 0, 0, 0)$ of the tetrahedron of reference at the inflexion point O ; then $c_0 = a_0 = b_0 = 0$. The equations of the tangent line of C at O now are $y = a_1x$, $z = b_1x$. If the edge $y = z = 0$ of the tetrahedron coincides with this tangent line, then $a_1 = b_1 = 0$. Since O is a point of inflexion, the tangent line must have precisely three-point contact with C at O and hence $a_2 = b_2 = 0$ and not both a_3 and b_3 are zero.

Since the curve C has an inflexion at O , any plane through the tangent at O will meet C in three consecutive points at O . Hence the osculating plane of C at O is indeterminate. However, the plane whose equation is $a_3z - b_3y = 0$ hyperosculates C at O . Since not both a_3 and b_3 are zero, the assumption that $a_3 \neq 0$ can be made without loss of generality. Let us choose the face $z = 0$ to coincide with this plane; then $b_3 = 0$. We shall assume that the hyperosculating plane has no higher than four-point contact with C at O since we do not wish the point O to be a singular point in any other sense than that of an inflexion; this means that $b_4 \neq 0$. For convenience, we shall replace a_3 by a and b_4 by b .

Next let us consider the quadric surface which osculates C at O . To find the equation of this quadric it is sufficient to write the most general equation of the second degree in x , y , and z and to demand that this equation be satisfied in x up to and including terms in x^8 . The result of this computation is

$$By^2 + Dz^2 + I\left[z - \frac{b}{a}xy + \left(\frac{a_4}{a} - \frac{b_5}{b}\right)xz\right] + Fyz = 0,$$

where B , D , I , and F are homogeneous parameters and have the following values:

$$(3) \quad B = \frac{1}{a^2} \cdot I\left[\frac{b}{a}a_5 - b_6 - \left(\frac{a_4}{a} - \frac{b_5}{b}\right)b_5\right],$$

$$(4) \quad F = -\frac{1}{ab} \cdot I\left[b_7 - \frac{b}{a}a_6 + \left(\frac{a_4}{a} - \frac{b_5}{b}\right)b_6\right] - 2aa_4B,$$

$$(5) \quad \begin{aligned} D &= -\frac{1}{b^2} \cdot I\left[b_8 - \frac{b}{a}a_7 + \left(\frac{a_4}{a} - \frac{b_5}{b}\right)b_7\right] \\ &\quad - B(a_4^2 + 2aa_5) - F(ab_5 + a_4b). \end{aligned}$$

Evidently $I \neq 0$ for noncomposite six-point quadrics.

If a point on this quadric is chosen as the vertex $(0, 0, 0, 1)$ of the tetrahedron, then $D = 0$. In homogeneous coördinates the equation of the tangent plane of the osculating quadric at the vertex $(0, 0, 0, 1)$ is

$$Ix_1 + I\left(\frac{a_4}{a} - \frac{b_5}{b}\right)x_2 + Fx_3 = 0.$$

Let us make the face $x_1 = 0$ coincide with this plane. Then $a_4/a = b_5/b$ and $F = 0$.

The hyperosculating plane $z = 0$ intersects the osculating quadric in two lines, the inflexional tangent of C at O , $y = z = 0$, and another line whose equations are $z = 0$ and $bIx - aBy = 0$. If the edge $x = z = 0$ coincides with this line, then $B = 0$. If we now choose a point on the osculating quadric as the unit point $U(1, 1, 1, 1)$, we have $a = b$. Thus the following conclusion is reached:

THEOREM 1.2. *By a suitable choice of the projective coördinate system the equation of the quadric surface osculating a space curve C at an inflexion point O can be reduced to the form $z - xy = 0$.*

In view of the conditions imposed on the coördinate system the equations of the curve C now are

$$(6) \quad \begin{aligned} y &= ax^3 + a_4x^4 + a_5x^5 + a_6x^6 + a_7x^7 + a_8x^8 + a_9x^9 + \cdots, \\ z &= ax^4 + a_4x^5 + a_5x^6 + a_6x^7 + a_7x^8 + b_9x^9 + \cdots \end{aligned} \quad (a \neq 0).$$

There are four conditions left which may be imposed on the coördinate system. Two of these conditions will be used to locate the edge $x = y = 0$ through the point O and two will be used to locate the unit point on the osculating quadric.

It will be useful to consider the cubic cone with vertex at O which osculates C at O . To find the equation of this cone, it is sufficient to write the most general homogeneous cubic equation in x , y , and z and demand that the expansions (6) for C satisfy this equation identically in x through terms in x^{12} . The result of this computation is

$$(7) \quad a^3y^3 - a^4xz^2 - a^2a_4y^2z + (aa_4^2 - a^2a_5)yz^2 + (2aa_4a_5 - a^2a_6 - a_4^3)z^3 = 0.$$

This cubic cone has an interesting property which is stated in the following theorem:

THEOREM 1.3. *The cubic cone osculating a space curve C at an inflexion point O is a cuspidal cubic cone with the tangent line of C at O as the cuspidal generator of the cone and with the hyperosculating plane of C at O as the cuspidal plane of the cone.*

Let us choose a point on this cuspidal cubic cone as the vertex $(0, 0, 0, 1)$ of the tetrahedron. This gives the condition

$$(8) \quad a^2a_6 - 2aa_4a_5 + a_4^3 = 0.$$

The cubic cone (7) has a Hessian cone whose equation is $z^2(3ay - a_4z) = 0$. Since the osculating cubic cone is a cuspidal cubic cone, it has an inflexional generator which lies on the Hessian as well as on the cuspidal cone. The equations of this inflexional generator are

$$x = a_4(7a_4^2 - 9aa_5)z/(27a^4), \quad y = a_4z/(3a).$$

If the edge $x = y = 0$ is made to coincide with this line, then $a_4 = 0$. Since $a_4 = 0$, condition (8) becomes $a_6 = 0$.

It remains only to determine the position of the unit point on the osculating quadric. The equation of the tangent plane of the cuspidal cubic cone (7) along the inflexional generator is $a^2x + a_5y = 0$. Let us choose the unit point as a point in this plane so that $a_5 = -a^2$. When this expression for a_5 and $a_4 = a_6 = 0$ are substituted in equation (7), the following conclusion is evident:

THEOREM 1.4. *By suitable choice of the coördinate system the equation of the cuspidal cubic cone osculating a space curve C at an inflexion point O can be put in the form $y^3 - axz^2 + ayz^2 = 0$, where the equation of its Hessian is $yz^2 = 0$.*

There is now one condition left which may be imposed on the unit point. With this in mind let us consider a quartic cone having the edge $x = y = 0$ as a double generator and the edge $x = z = 0$ as a single generator and osculating C at O . The equation of such a cone will be found by writing the most general equation of the fourth degree homogeneous in x, y , and z , with the terms in z^4, xz^3 , and yz^3 missing since the edge $x = y = 0$ is a double generator and the term y^4 missing because the edge $x = z = 0$ is a single generator and demanding that the expansions (6) with $a_4 = a_6 = 0$ and $a_5 = -a^2$ satisfy this equation through terms in x^{14} . This computation gives the equation of the quartic cone as

$$xyz^3 - ax^2z^2 + axyz^2 + [a - a_7/(a^2)]y^2z^2 = 0.$$

Let us suppose that the unit point is on this cone. Then $a_7 = a^2(a + 1)$ and the equation of the quartic cone reduces to

$$(9) \quad xy^3 - ax^2z^2 + axyz^2 - y^2z^2 = 0.$$

Since all of the fifteen conditions at our disposal have been imposed on the coördinate system, it is well determined and is entirely defined geometrically. When all of the conditions above are satisfied, series (6) reduce to series (2) and Theorem 1.1 is proved.

1.2. Osculating surfaces. The canonical expansions (2) and other results of the preceding section will now be used to investigate further properties of the space curve C in the neighborhood of an inflexion point O on C . This section is devoted to a study of some of the surfaces osculating C at O .

Let us first investigate the nature of quadrics osculating C at O . The six-parameter family of quadrics having three-point contact with C at O is represented by the equation

$$(10) \quad Ay^2 + Bz^2 + Dxy + Exz + Fyz + Hy + Iz = 0,$$

where the capital letters denote homogeneous parameters. Some of the properties of these quadrics are stated in the following theorem:

THEOREM 1.5. *The inflexional tangent of C at O , $y = z = 0$, lies on every three-point quadric of C at O . If $DI - EH \neq 0$ in (10), then the hyperosculating plane of C at O , $z = 0$, is tangent at the point $(D, -H, 0, 0)$ to these quadrics.*

If the quadrics represented by equation (10) have four-point contact with C at O , then $H = 0$. Thus the following conclusion is reached:

THEOREM 1.6. *The hyperosculating plane of C at O is tangent, at the point O , to all four-point quadrics of C at O .*

The equation of the two-parameter family of quadrics having seven points in common with C at O is

$$(11) \quad Bz^2 + D(xy - z) + Fyz = 0,$$

where B , D , and F are homogeneous parameters. We see that the edge $x = z = 0$ lies on all these quadrics. The condition that they be cones with vertices at O is $D^4 = 0$.

THEOREM 1.7. *Each quadric cone of the one-parameter family of quadric cones having seven-point contact with C at an inflexion point O is composite, being composed of two planes one of which is the hyperosculating plane of C at O .*

If equation (11) represents the one-parameter family of eight-point quadrics of C at O , then $F = 0$. If these quadrics are cones with vertices at O , then $D = 0$. Hence it is clear that the following theorem is true:

THEOREM 1.8. *The only eight-point quadric cone with vertex at O and osculating C at O is the cone composed of the hyperosculating plane of C at O counted twice.*

In the last section the equation of the osculating, or nine-point, quadric of C at O was found to be $z - xy = 0$. When the canonical expansions (2) are substituted in the left member of this equation, the result is

$$z - xy = (b_9 - a_8)x^9 + (b_{10} - a_9)x^{10} + (b_{11} - a_{10})x^{11} + \dots$$

Therefore, we have a geometric meaning for the equation $b_9 = a_8$; for if this equation is true, the osculating quadric has ten-point contact with C at O . If also $b_{10} = a_9$ in (2), the osculating quadric will have eleven-point contact with C at O , and so on.

Next let us consider some of the cubic cones which osculate C at O . The equation of the eight-parameter family of cubic cones having five-point contact with C at O is

$$(12) \quad By^3 + Az^3 + Dx^2y + Exy^2 + Fxyz + Gx^2z + Hxz^2 + Iy^2z + Jyz^2 = 0,$$

where the capital letters denote homogeneous parameters. The following conclusion is immediate:

THEOREM 1.9. *The tangent line, $y = z = 0$, of a space curve C at an inflexion point O lies on all five-point cubic cones of C at O .*

The conditions that equation (12) represent the six-parameter family of seven-point cubic cones are $D = G = 0$. Since the terms in x^3 and x^2 are all missing, the following theorem is evident:

THEOREM 1.10. *The tangent line of C at O is a double line on all the seven-point cubic cones of C at O .*

The conditions that equation (12) represent the four-parameter family of nine-point cubic cones of C at O are $D = E = F = G = 0$. The following conclusion is immediate:

THEOREM 1.11. *The tangent line of C at O is a cuspidal generator of all the cubic cones having nine-point contact with C at O and the cuspidal plane along this tangent line is the hyperosculating plane of C at O .*

In the last section the equation of the osculating cubic cone, which has thirteen-point contact with C at O , was found to be

$$(13) \quad y^3 - axz^2 + ayz^2 = 0.$$

Three of the four vertices of the tetrahedron of reference lie on this cone. The fourth vertex has an interesting property which is stated in the following theorem:

THEOREM 1.12. *The polar plane of the vertex $(0, 0, 1, 0)$ with respect to the cubic cone (13) is the face $y = 0$, which is also the tangent plane of the osculating quadric at the vertex $(0, 1, 0, 0)$. The polar quadric cone of this vertex with respect to the cone (13) is composite, being composed of two planes represented by the equation $3y^2 + az^2 = 0$.*

Two other properties of the cuspidal cone (13) are given in the following theorem:

THEOREM 1.13. *The equation of the inflexional tangent plane of the cuspidal cubic cone is $x - y = 0$. The polar quadric cone of the vertex $(0, 0, 0, 1)$ with respect to the cubic cone is composed of the inflexional tangent plane and the hyperosculating plane of C at O .*

If the power series (2) for y and z are substituted in the left member of equation (13), the result is

$$y^3 - axz^2 + ayz^2 = a^4x^{13} + a^2(3a_8 - 2b_9)x^{14} + \dots$$

Thus we see that the osculating cubic cone never has more than thirteen-point contact with C at the inflexion point O (not otherwise singular) since $a \neq 0$.

The last surface we shall mention here is the quartic cone (9) which osculates

C at O and which has already been used in deducing the canonical expansions (2). Some of the properties of this cone are stated below:

THEOREM 1.14. *All the vertices of the tetrahedron and the unit point lie on the quartic cone (9). The tangent line of C at O is a cuspidal generator and the cuspidal plane along this generator is the hyperosculating plane of C at O . Along the generator $x = z = 0$ the tangent plane is the face $x = 0$. Along the double generator $x = y = 0$ the tangent planes are represented by the equation $ax^2 - axy + y^2 = 0$. The equation of the tangent plane along the generator $x = y = z$ through the unit point is $(1 - a)x + (1 + a)y - 2z = 0$.*

The faces $x = 0$ and $y = 0$ of the tetrahedron are bitangent planes of the quartic (9) since $x = 0$ cuts the cone in the generators $x = y = 0$ and $x = z = 0$ each counted twice and $y = 0$ cuts the cone in the edges $x = y = 0$ and $y = z = 0$ each counted twice.

1.3. Plane sections of the tangent developable. In this section we shall consider the plane sections of the tangent developable of C , in the neighborhood of the inflexion point O , made by the faces $z = 0$, $x = 0$, and $y = 0$, respectively. If (x, y, z) is a point on C in the neighborhood of O , then it is a well-known fact that the parametric equations of the tangent line of C at (x, y, z) are

$$(14) \quad X = x + r, \quad Y = y + ry', \quad Z = z + rz',$$

where r is the parameter, the accents denote differentiation with respect to x , and (X, Y, Z) is a variable point on the tangent line. As the point (x, y, z) moves along C , equations (14) represent the tangent developable of C in the neighborhood of O .

In order to study the section of this tangent developable by the plane $Z = 0$, we set $Z = 0$ in (14). This gives a value for r in terms of z and z' . From equations (2) for z , r can be expressed in terms of x . Thus X and Y can be expressed as power series in x . By eliminating x , we find the following equations for the plane section:

$$Y = \frac{1}{4} \left(\frac{4}{3} \right)^3 a X^3 - \frac{1}{4} \left(\frac{4}{3} \right)^7 a^2 \left(\frac{a}{12} + 1 \right) X^7 + \frac{1}{4} \left(\frac{4}{3} \right)^8 \left(\frac{5}{2} b_9 - 4a_8 \right) X^8 + \cdots, \\ Z = 0 \quad (a \neq 0).$$

The nature of these equations implies the following theorem:

THEOREM 1.15. *The intersection of the tangent developable (14) by the hyperosculating plane of C at O is a plane curve with an inflexion at O and with the same inflexional tangent as C . This curve has precisely three-point contact with C at O .*

By a method similar to the one used above the equations of the section of the tangent developable made by the face $X = 0$ are found to be

$$(15) \quad Z = \frac{3Y^{4/3}}{2(-2a)^{1/3}} - \frac{3}{4} Y^2 + \cdots, \quad X = 0.$$

From the nature of these equations, we draw the conclusion:

THEOREM 1.16. *The plane section (15) has a triple point $[1; 215, 7^\circ]$ at O with coincident triple point tangents, the tangent being the edge $X = Y = 0$.*

Similarly the equations of the section made by the face $Y = 0$ are found to be

$$(16) \quad Z = -\frac{1}{3}\left(\frac{2}{3}\right)^4 aX^4 - \frac{1}{3}\left(\frac{2}{3}\right)^6 a^2X^6 + \cdots, \quad Y = 0.$$

The following conclusion is immediate:

THEOREM 1.17. *The section (16) is a curve having an undulation point at O , having the same tangent at O as C , and having three-point contact with C at O .*

1.4. Projections of the curve. In this section we study the projections of the space curve C , represented by equations (2), from certain points onto the faces of the tetrahedron of reference. First, we shall consider the projection of C from a point (α, β, γ) , with $\gamma \neq 0$, onto the hyperosculating plane of C at O . The point (α, β, γ) will then be chosen in the face $x = 0$, in the face $y = 0$ and finally on the edge $x = y = 0$. A similar study of the projections on the faces $y = 0$ and $x = 0$ will be made.

In order to find the equations of the projection of C in the neighborhood of the inflexion point O from a point (α, β, γ) , $\gamma \neq 0$, we consider the line joining the point (α, β, γ) and a variable point (x, y, z) on C . The parametric equations of this line are

$$(17) \quad X = x + (\alpha - x)t, \quad Y = y + (\beta - y)t, \quad Z = z + (\gamma - z)t,$$

where t is the parameter and X, Y , and Z are the coördinates of any point on the line. The intersection of the line (17) with the face $Z = 0$ is then the projection of C onto the hyperosculating plane of C at O . If we set $Z = 0$ in (17), solve for t and substitute the value of z from equations (2), we have a value for t in terms of x . We set this value for t in the first two of equations (17), solve for x in terms of X and then substitute in the expression for Y , where the value of y is found from (2). The result is

$$(18) \quad Y = aX^3 - a\left(\frac{\beta}{\gamma}\right)X^4 - a^2X^5 + a^2\left[3\left(\frac{\alpha}{\gamma}\right) + \left(\frac{\beta}{\gamma}\right)\right]X^6 + \cdots, \quad Z = 0,$$

which are the equations of the projection of C . The following theorem is immediate:

THEOREM 1.18. *The projection (18) is a curve with an inflexion at O , with the same tangent line at O as C , and having four-point contact with C at O .*

If the point (α, β, γ) is chosen in the face $X = 0$ but not on the edge $X = Z = 0$, the conclusions in the above theorem are not changed. However, if the point (α, β, γ) is chosen in the face $Y = 0$ but not on the edge $Y = Z = 0$, then $\beta = 0$ in equations (18) and the series for Y coincides with the series for y in equations (2) through terms in x^5 . Thus we see that the cone represented by the series

for Y in (18), with $\beta = 0$, has six-point contact with C at O . The results in the above theorem are still unchanged when $\beta = 0$.

If we set $\alpha = \beta = 0$ in equations (18), it is clear that we obtain the equations of the projection of C from a point, distinct from O , on the edge $X = Y = 0$ onto the hyperosculating plane of C at O . The results in Theorem 1.18 are unaltered by this choice of the point (α, β, γ) . However, the cone represented by the first of equations (18), with $\alpha = \beta = 0$, has precisely seven-point contact with C at O for in this case the coefficient of x^7 reduces to $a^2(a + 1 - 2/\gamma)$.

Next we shall consider the projection of C from a fixed point (α, β, γ) , with $\beta \neq 0$, in space, onto the face $Y = 0$. If we follow the same procedure used in the preceding case, the equations of this projection are found to be $Y = 0$ and

$$(19) \quad Z = -a \frac{\gamma}{\beta} X^3 + aX^4 + a^2 \left(\frac{\gamma}{\beta} - \frac{3\alpha\gamma}{\beta^2} \right) X^5 + a^2 \left(\frac{2\gamma}{\beta^2} + \frac{4\alpha}{\beta} - 1 \right) X^6 + \dots$$

($\beta \neq 0$).

The nature of this equation implies the following conclusion:

THEOREM 1.19. *The projection of C , in the neighborhood of O , from a point (α, β, γ) , $\beta \neq 0$, onto the face $Y = 0$ is a curve having three-point contact with C at O and having the same tangent at O as C . If (α, β, γ) does not lie in the hyperosculating plane of C at O , the projection has a simple inflexion at O ; if it does, the curve has an undulation point at O .*

If the point (α, β, γ) lies in the face $X = 0$, but not on the edge $X = Y = 0$, then $\alpha = 0$ in equation (19). However, the first two terms are not affected by setting $\alpha = 0$ and hence the conclusions in the above theorem are unchanged. If the point (α, β, γ) is chosen as a point, distinct from O , on the edge $X = Z = 0$, then $\alpha = \gamma = 0$ in equation (19) and the first term in the expansion is aX^4 . Hence the projection also has an undulation point at O with the same tangent at O as C . The plane curve itself has three-point contact with C at O , but the cone represented by the series for Z has precisely seven-point contact with C at O since the coefficient of X^7 , when computed, is found to be $-3a^2/\beta$.

Next let us project the curve C from a fixed point (α, β, γ) , with $\alpha \neq 0$, onto the face $X = 0$. The equations of this projection are

$$(20) \quad Z = \frac{\gamma}{\beta} Y + a \frac{\alpha^3 \gamma}{\beta^4} Y^3 + a \left(\frac{2\alpha^3 \gamma}{\beta^5} + \frac{\alpha^4}{\beta^4} \right) Y^4 + \dots, \quad X = 0 \quad (\alpha\beta \neq 0),$$

where it is assumed that (α, β, γ) does not lie in the face $Y = 0$. The nature of this projection is given in the following theorem:

THEOREM 1.20. *The projection (20) is a curve having an inflexion or a singularity of higher degree at O with $\beta Z - \gamma Y = 0$ and $X = 0$ as the equations of the tangent at O . In case the point (α, β, γ) lies in the face $Z = 0$ but not on either the edge $Y = Z = 0$ or $X = Z = 0$, the projection onto the face $X = 0$ has an undulation point at O with the edge $X = Z = 0$ as the undulational tangent.*

Let us now choose (α, β, γ) in the face $Y = 0$ and project C onto the face $X = 0$. The usual methods of computation give as the equations of this projection

$$(21) \quad Y = -a \frac{\alpha^3}{\gamma^3} Z^3 - 2a \frac{\alpha^3}{\gamma^4} Z^4 + a \frac{\alpha^3}{\gamma^5} (a\alpha^2 - 3)Z^5 + \cdots, \quad X = 0$$

($\alpha\gamma \neq 0$),

where we have assumed $\gamma \neq 0$ since the case $\gamma = 0$ has already been treated. The following conclusion is immediate:

THEOREM 1.21. *The projection (21) is a curve having an inflexion at O and having the edge $X = Y = 0$ as tangent line at O .*

Finally, if C is projected from a point distinct from O on the tangent of C at O onto the face $X = 0$, the equations of the projection are

$$(22) \quad Z = \frac{1}{a^{1/3}} Y^{4/3} - \frac{1}{3a^{2/3}\alpha} Y^{5/3} + \cdots, \quad X = 0 \quad (\alpha \neq 0).$$

From the nature of these equations the following conclusion may be stated:

THEOREM 1.22. *The projection (22) is a curve with a triple-point $[1; 215, 7^\circ]$ at O with the triple-point tangents coinciding in the edge $X = Y = 0$.*

Chapter II. Planar Point

2.1. Canonical power-series expansions. As in §1.1 we shall consider an analytic curve C which can be represented by the power-series expansions (1) in the neighborhood of an ordinary point O on C . Henceforth the hypothesis will be made that O is a planar point; that is, a point at which the unique osculating plane of C at O has precisely four-point contact instead of the usual three-point contact. The purpose of this section will then be to prove the following theorem:

THEOREM 2.1. *By suitable choice of the projective coördinate system the power series representing a space curve C in the neighborhood of a planar point O (not otherwise singular) can be reduced to the following canonical form:*

$$(23) \quad \begin{aligned} y &= x^2 - x^5 + x^6 - b_6x^7 + (b_6 - b_7)x^8 + a_9x^9 + \cdots, \\ z &= x^4 + b_6x^6 + b_7x^7 + b_8x^8 + b_9x^9 + \cdots. \end{aligned}$$

As in §1.1 we first simplify expansions (1) by choosing the vertex $(1, 0, 0, 0)$ of the tetrahedron at the point O so that $c_0 = a_0 = b_0 = 0$. Also we shall choose the edge $y = z = 0$ along the tangent of C at O so that $a_1 = b_1 = 0$.

The equation of the osculating plane of C at O is $a_2z - b_2y = 0$. Not both a_2 and b_2 are zero unless the tangent line has more than two-point contact with C at O . We wish to exclude this possibility here so we shall assume $a_2 \neq 0$,

which can be done without loss of generality. Then let us choose the face $z = 0$ of the tetrahedron as the osculating plane of C at O so that $b_2 = 0$. Since the osculating plane of C at O has precisely four-point contact with C at O , it follows that $b_3 = 0$ and $b_4 \neq 0$. Thus the equations (1) of C have been reduced to the form

$$(24) \quad \begin{aligned} y &= ax^2 + a_3x^3 + a_4x^4 + a_5x^5 + \cdots, \\ z &= bx^4 + b_5x^5 + \cdots \end{aligned} \quad (ab \neq 0),$$

where a_2 has been replaced by a and b_4 by b for convenience.

In choosing the coördinate system so as to reduce expansions (24) to a canonical form we shall use the osculating quadric of C at O . The equation of this quadric is

$$(25) \quad H(y - ax^2 - a_3xy/a) + By^2 + Dz^2 + Exz + Fyz + Iz = 0,$$

where the capital letters denote homogeneous parameters and have the values:

$$I = H[a_3^2/(ab) - a_4/b] - Ba^2/b,$$

$$E = H[a_3a_4/(ab) - a_5/b] - 2Baa_3/(b) - Ib_5/b,$$

$$F = \frac{1}{ab} [H(a_3a_5/(a) - a_6) - B(a_3^2 + 2aa_4) - Eb_5 - Ib_6],$$

$$H(a_7 - a_3a_6/a) + B(2aa_5 + 2a_3a_4) + Eb_6 + F(ab_5 + a_3b) + Ib_7 = 0,$$

$$D = \frac{1}{b^2} [H(a_3a_7/(a) - a_8) - B(a_4^2 + 2aa_6 + 2a_3a_5) - Eb_7 - F(ab_6 + a_3b_5 + a_4b) - Ib_8].$$

The stationary osculating plane $z = 0$ cuts the osculating quadric in a conic, whose equations are

$$H(y - ax^2 - a_3xy/a) + By^2 = 0, \quad z = 0.$$

Let us choose the vertex $(0, 0, 1, 0)$ of the tetrahedron as a point on this conic so that $B = 0$. In homogeneous coördinates the equation of the tangent plane of the osculating quadric at the vertex $(0, 0, 1, 0)$ is

$$(26) \quad Hx_1 - a_3Hx_2/(a) + Fx_4 = 0.$$

This tangent plane cuts the osculating quadric in two generators. Let the unit point be chosen as a point on one of these generators, so that $F = a_3H/(a) - H$ and $E = aH - D - I$.

The tangent line $y = z = 0$ of C at O cuts the plane (26) in the point whose coördinates satisfy the conditions

$$Hx_1 - a_3Hx_2/a = 0, \quad x_3 = 0, \quad x_4 = 0.$$

Equation (25) shows that, when $B = 0$, $H \neq 0$ for a noncomposite osculating quadric. Then the vertex $(0, 1, 0, 0)$ can be chosen to coincide with this point, so that $a_3 = 0$.

The equation of the tangent plane of the osculating quadric at the vertex $O(1, 0, 0, 0)$ is $Hy + Iz = 0$. Since $H \neq 0$, this tangent plane is distinct from the stationary osculating plane $z = 0$ and may be taken as the face $y = 0$ of the tetrahedron so that $I = 0$. The face $y = 0$ cuts the quadric in two generators whose equations are $aHx^2 + (D - aH)xz - Dz^2 = 0$ and $y = 0$. Let the edge $x = y = 0$ be chosen along one of these generators; then $D = 0$. The vertex $(0, 0, 0, 1)$ is a point on this generator but its exact position on the line has not yet been located. The equation of the tangent plane of the osculating quadric at this point is $ax - y = 0$. This plane cuts the quadric in two generators. Let us choose the unit point on one of these generators, so that $a = 1$. Now it is possible to state the following theorem:

THEOREM 2.2. *By suitable choice of the coördinate system the osculating quadric at a planar point O of a space curve C can be represented by the equation $y - x^2 + xz - yz = 0$.*

Of the fifteen conditions which we are free to impose on the coördinate system thirteen have already been imposed. There are then two conditions left, one on the vertex $(0, 0, 1, 0)$, which is at present restricted to be on a certain conic, and the other on the vertex $(0, 0, 0, 1)$, which is at present restricted to be on a certain generator of the osculating quadric.

In order to impose these two conditions, we shall use the noncomposite cuspidal cubic cone with vertex at O , having the edge $x = y = 0$ as cuspidal generator, and having eight-point contact with C at O . The equation of this cone is

$$(27) \quad 4b^3y^3 - 4b^2x^2z + 4bb_5xyz - b_5^2y^2z = 0,$$

and that of its cuspidal plane is $2bx - b_5y = 0$. Let us fix the position of the vertex $(0, 0, 1, 0)$ by choosing the face $x = 0$ to coincide with this plane; then $b_5 = 0$. Also let us choose the vertex $(0, 0, 0, 1)$ so that one of the generators of the osculating quadric through this point pierces the cuspidal cubic cone (27) in the unit point. With this choice the unit point is on the cuspidal cubic cone so that $b = 1$. When account is taken of these two conditions in equation (27), the following theorem is immediate:

THEOREM 2.3. *By suitable choice of the coördinate system the equation of the cuspidal cubic cone with vertex at O having the edge $x = y = 0$ as cuspidal generator and having eight-point contact with C at O can be reduced to the form*

$$(28) \quad y^3 - x^2z = 0.$$

The coördinate system has now been well determined and entirely defined geometrically. When all the conditions found above have been substituted in expansions (24), Theorem 2.1 is proved.

2.2. Osculating surfaces. The canonical expansions (23) and other results of the last section will now be used to study the neighborhood of a planar point O on a space curve C . In this section we shall study some of the surfaces osculating C at O .

The equation of the two-parameter family of quadrics having seven-point contact with C at O is

$$(29) \quad H(x_1x_3 - x_2^2 + x_2x_4 - x_3x_4) + B(x_3^2 - x_1x_4 + b_6x_3x_4) + Dx_4^2 = 0,$$

where H , B , and D are homogeneous parameters. The conditions that this equation represent a quadric cone with vertex at O are $H = B = 0$. Therefore the following conclusion is evident:

THEOREM 2.4. *The only quadric cone having its vertex at O and having seven-point contact with C at O is composed of the stationary osculating plane of C at O counted twice.*

It is a well-known theorem that all quadrics passing through seven points pass also through an eighth point. In the case of the quadrics represented by equation (29) this point is readily found to coincide with the point O itself.

Next let us consider some of the cubic cones osculating C at O . The equation of the eight-parameter family of cubic cones with vertex O and having four-point contact with C at O is

$$(30) \quad Ay^3 + Bz^3 + Dx^2y + Ex^2z + Fxy^2 + Gxz^2 + Hyz^2 + Iy^2z + Jxyz = 0,$$

where the capital letters denote homogeneous parameters. The nature of equation (30) shows us that the tangent of C at O , $y = z = 0$, is a generator of all these cones.

If equation (30) represents a cubic cone having the edge $x = y = 0$ as a cuspidal generator and having eight-point contact with C at O , it reduces to $y^3 - x^2z = 0$, which is the equation (28) of the cubic cone used in deducing the canonical expansions (23). The Hessian of this cubic cone is composite, being composed of the cuspidal plane $x = 0$ counted twice and the face $y = 0$, which is also the tangent plane of the osculating quadric at O . When the equation of the cone is solved simultaneously with that of its Hessian, an interesting property may be noted:

THEOREM 2.5. *The inflexional generator of the cuspidal cubic cone (28) is the tangent line of C at O and the tangent plane of the cone along the inflexional generator is the stationary osculating plane of C at O .*

Some of the other properties of this cone are stated in the following theorem:

THEOREM 2.6. *The polar plane of the vertex $(0, 0, 1, 0)$ with respect to the cuspidal cubic cone (28) is the face $y = 0$. The polar quadric cone of the vertex $(0, 1, 0, 0)$ with respect to the cone (28) is composite, being composed of the cuspidal plane of the cone and the stationary osculating plane of C at O . The polar quadric cone of the vertex $(0, 0, 1, 0)$ is also composite, being composed of the face $y = 0$ counted twice.*

When the canonical expansions (23) are substituted in the left member of the equation (28), the result is

$$y^3 - x^2z = -b_6x^8 - (b_7 + 3)x^9 + \cdots$$

Hence we see that, if $b_6 = 0$ in expansions (23), the cuspidal cubic cone (28) hyperosculates C at O and thus a geometric meaning is attached to the vanishing of b_6 .

If we demand that equation (30) represent the cubic cone osculating C at O , the following theorem is the result:

THEOREM 2.7. *The osculating cubic cone of a space curve C at a planar point O is composite, being composed of the stationary osculating plane of C at O counted three times.*

If we demand that equation (30) represent a cubic cone having nine-point contact with C at O and having the edge $x = y = 0$ as a double generator, we find that the equation of this cone is $y^3 - x^2z + b_6y^2z = 0$. It follows that, if $b_6 = 0$, the double generator $x = y = 0$ is a cuspidal generator of the cone, and the cone is the one used in deducing the canonical expansions (23).

2.3. Plane sections of the tangent developable. This section is devoted to a study of plane sections of the tangent developable of C in the neighborhood of the planar point O made by the faces of the tetrahedron. This study is similar to the one of §1.3 where O was assumed to be an inflexion point. As in that section the tangent developable of C is defined by equations (14).

If we follow the procedure of §1.3 using expansions (23) to represent C in the neighborhood of the planar point O , the equations of the section made by the stationary osculating plane of C at O are found to be

$$(31) \quad Y = \frac{1}{2}\left(\frac{4}{3}\right)^2 X^2 + \frac{1}{12}\left(\frac{4}{3}\right)^4 b_6 X^4 + \cdots, \quad Z = 0.$$

The following conclusion is evident:

THEOREM 2.8. *The plane curve (31) has the same tangent at O as C and has two-point contact with C at O .*

Similarly, the equations of the section of the tangent developable made by the face $Y = 0$ are

$$(32) \quad Z = -16X^4 - 128b_6X^6 + \cdots, \quad Y = 0.$$

The following theorem is immediate:

THEOREM 2.9. *The curve (32) has two-point contact with C at O , an undulation point at O , and the same tangent at O as C .*

When the equations of the intersection of the tangent developable of C and the face $X = 0$ are computed, the results are

$$(33) \quad Z = -3Y^2 + 5b_6Y^3 - 6i(b_7 + 4)Y^{7/2} + \cdots, \quad X = 0.$$

Thus we have the following conclusion:

THEOREM 2.10. *The section (33) is a curve having a tacnode-cusp [1; 216, (4)] at O and having for tangent at O the edge $X = Z = 0$.*

2.4. Projections of the curve. In this section the projections of the space curve C , represented by equations (23), in the neighborhood of the planar point O , from various points onto the faces of the tetrahedron of reference will be studied. We shall use here the notations of §1.4 and follow the procedure of that section in finding the equations of the projections.

First, we study the projection of C from a point (α, β, γ) , $\gamma \neq 0$, onto the stationary osculating plane of C at O . The equations of this projection are

$$(34) \quad \begin{aligned} Y = X^2 - \frac{\beta}{\gamma} X^4 + \left(2\frac{\alpha}{\gamma} - 1\right) X^5 + \left(1 - \frac{\beta}{\gamma} b_6 - \frac{1}{\gamma}\right) X^6 \\ + \left(2\frac{\alpha}{\gamma} b_6 - 4\frac{\alpha\beta}{\gamma^2} - b_6 - \frac{\beta}{\gamma} b_7\right) X^7 + \cdots, \quad Z = 0 \quad (\gamma \neq 0). \end{aligned}$$

These equations justify the following theorem:

THEOREM 2.11. *The projection (34) is a curve having four-point contact with C at O and having the same tangent at O as C .*

The leading term of equation (34) is not changed when $\alpha = 0$ or $\beta = 0$. Hence the conclusions of Theorem 2.11 hold if (α, β, γ) , $\gamma \neq 0$, is a point in the face $X = 0$, in the face $Y = 0$, or on the edge $X = Y = 0$.

The equations of the projection of C onto the face $Y = 0$ from a point (α, β, γ) , with $\beta \neq 0$, are

$$(35) \quad \begin{aligned} Z = -\frac{\gamma}{\beta} X^2 - 2\frac{\alpha\gamma}{\beta^2} X^3 + \left(1 + \frac{\gamma}{\beta^2} - 5\frac{\alpha^2\gamma}{\beta^3}\right) X^4 \\ + \left(4\frac{\alpha}{\beta} + 6\frac{\alpha\gamma}{\beta^3} + \frac{\gamma}{\beta} - 14\frac{\alpha^3\gamma}{\beta^4}\right) X^5 + \cdots, \quad Y = 0 \quad (\beta \neq 0). \end{aligned}$$

From these equations the theorem below follows:

THEOREM 2.12. *The projection (35) has two-point contact with C at O and has the same tangent at O as C .*

The results of Theorem 2.12 are not changed if $\alpha = 0$ in equations (35). On the other hand, if $\gamma = 0$, the following theorem is the result:

THEOREM 2.13. *The projection of C , in the neighborhood of O , from a point in the stationary osculating plane of C at O , but not on the tangent of C at O , onto the face $Y = 0$ is a curve having an undulation point at O , having the same tangent at O as C , and having two-point contact with C at O .*

None of the properties in the above theorem is changed if (α, β, γ) is a point, distinct from O , on the edge $X = Z = 0$.

Finally, let us investigate the nature of the projections of C from the point (α, β, γ) , with $\alpha \neq 0$, onto the face $X = 0$. If we also assume $\beta \neq 0$, the equations of the projection are

$$(36) \quad Z = \frac{\gamma}{\beta} Y - \frac{\alpha^2 \gamma}{\beta^3} Y^2 + \frac{\alpha^2 \gamma}{\beta^4} \left(2 \frac{\alpha^2}{\beta} - 1 \right) Y^3 + \cdots, \quad X = 0 \quad (\alpha\beta \neq 0).$$

The point O is an ordinary point on the projection (36) if $\gamma \neq 0$ and the equations of the tangent line at O are $\beta Z - \gamma Y = 0$ and $X = 0$.

If $\gamma = 0$ in equations (36), they represent the projection of C from a point in the stationary osculating plane of C at O but not on the tangent of C at O nor the edge $X = Z = 0$. These equations are

$$(37) \quad Z = \frac{\alpha^4}{\beta^4} Y^4 - \frac{\alpha^4}{\beta^6} (3\beta - 4\alpha^2) Y^5 + \cdots, \quad X = 0 \quad (\alpha\beta \neq 0),$$

where the coefficients of Y^4 and Y^5 in the series for Z have now been computed. The form of equations (37) implies the following theorem:

THEOREM 2.14. *The projection (37) is a curve having an undulation point at O and having the edge $X = Z = 0$ as the tangent line at O .*

Now let us project C from the point $(\alpha, 0, \gamma)$, $\alpha\gamma \neq 0$, onto the face $X = 0$. The equations of this projection are

$$Y = \frac{\alpha^2}{\gamma^2} Z^2 + \frac{\alpha^2}{\gamma^3} Z^3 + \frac{\alpha^2}{\gamma^4} Z^4 + \cdots, \quad X = 0 \quad (\alpha\gamma \neq 0).$$

The point O is an ordinary point on this projection and the tangent line of this curve at O is the edge $X = Y = 0$.

Finally, if the curve C is projected from a point distinct from O on the tangent line of C at O onto the face $X = 0$, the equations of the projection are

$$(38) \quad Z = Y^2 - \frac{1}{\alpha} Y^{5/2} + \cdots, \quad X = 0 \quad (\alpha \neq 0).$$

From the form of these equations the following theorem is justified:

THEOREM 2.15. *The projection (38) has a ramphoid cusp [1; 216, (2)] at O and has the edge $X = Z = 0$ as its tangent line at O .*

REFERENCE

1. G. SALMON, *A Treatise on the Higher Plane Curves*, Dublin, 1879.

CONNECTICUT COLLEGE.

